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روشی نو برای محاسبه عدد فرانزیوس

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$$A_k = \{a_0, a_1, \dots, a_k\}$$
$$N \quad (a_0, a_1, \dots, a_k) = 1$$
$$x_0, x_1, \dots, x_k \in N_o = N \cup \{0\}$$
$$N = a_0 x_0 + a_1 x_1 + \dots + a_k x_k$$

A New Approach for Calculating Frobenius Number

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Abstract

In old times some countries used to coin their coins in the bases of prime numbers and people, as an entertainment, were trying to find out "the most amount that can not be payed with the combination of the belonging coins". This number is known as the *Frobenius number*.

More precisely, let $A_k = \{a_0, a_1, \dots, a_k\}$ be a set of natural numbers such that $\{a_0, a_1, \dots, a_k\} = 1$. Then the natural number N has a representation in the bases of A_k , if there exist $\{x_0, x_1, \dots, x_k\} \in N_0 = N \cup \{0\}$ such that

$$N = x_0 a_0 + x_1 a_1 + \dots + x_k a_k.$$

Now, the largest number, which can not be represented in the above form is called the *Frobenius number*.

In the present paper we try to give a new technique for the calculation of the Frobenius number for certain sets.

Keywords: The Frobenius number, Number representation

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$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16\}$$

$$\{5, 7, 13\}$$

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$$A_k = \{a_0, a_1, \dots, a_k\}$$

:

$$(a_0, a_1, \dots, a_k) = 1$$

$$x_1, \dots, x_k$$

$$A_k$$

$$N$$

$$N = x_0 a_0 + x_1 a_1 + \dots + x_k a_k$$

$$g(A_k) = g(a_0, a_1, \dots, a_k)$$

$$1 \in A_k$$

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$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad A_k$$

[[] []

$$n = ax_0 + by_0 \quad n \quad x_0, y_0 \in \mathbb{Z} \\ (x, y) \in \mathbb{Z} \times \mathbb{Z}$$

$$\begin{aligned} x &= x_0 + bt \\ y &= y_0 + at \end{aligned} \quad , t \in \mathbb{Z}$$

$$0 \leq x_1 \leq b-1$$

$$.n = xa + yb$$

$$.n = x_1a + y_2b$$

$$n = xa + yb$$

$$x, y \in \mathbb{Z}$$

$$(x - x_0)a + (y - y_0)b = 0$$

$$(x - x_0)a = (y - y_0)b$$

$$.b | (x - x_0) \quad a | (y - y_0) \quad (a, b) = 1$$

$$x = x_0 + bt$$

$$y = y_0 + at$$

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 \ G. Hofmeister
 \ E. S. Selmer

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$$t = \left\lfloor \frac{x_0}{b} \right\rfloor \quad 0 \leq x_1 \leq b-1$$

$$A_2 = \{a, b\}$$

$$g(a, b) = ab - (a + b)$$

$$g = g(a, b)$$

b a

g

$$g = ab - a - b = a(b-1) - 1b$$

$$x_0, y_0 \in \mathbf{N}_0 = \mathbf{N} \cup \{0\} \quad y = -1 \quad x = b-1$$

$$g = ax_0 + by_0$$

$$t \in \mathbf{Z}$$

$$x_0 = x + bt$$

$$y_0 = y - at$$

$$t \leq -1$$

$$y_0 \geq 0$$

$$x_0 = b-1 + bt \leq b-1 - b = -1$$

$$N_0 \quad g$$

$$x_0 \leq -1$$

$$n \geq g = g(a, b)$$

$$x, y \in \mathbf{Z}$$

$$(a, b) = 1$$

b a

$$n = ax + by$$

$$x_0 = x + bt$$

$$t \in \mathbf{Z}$$

$$n = ax_0 + by_0$$

$$y_0 = y - at$$

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$$\begin{aligned}
 t = [y/a] \quad) \quad at \leq y & \quad t \\
 at < y - a & \quad y - a \leq at \quad .(\quad y/a \\
 & \quad t \quad a(t+1) = at + a < y \\
 n = ax_0 + by_0 > g(a,b) & \quad . y_0 = y - at \geq 0 \quad at \leq y \\
 ax_0 > ab - (a+b) - by_0 & = ab - (a+b) - b(y - at) \\
 & = b[at - (y - a)] - (a+b) \geq b - (a+b) = -a \\
 x_0 \geq 0 & \quad . x_0 \geq -1 \quad ax_0 > -a
 \end{aligned}$$

$$\begin{aligned}
 & \quad b \quad a \quad . . \\
 \frac{g+1}{2} & \quad \{a,b\} \quad 0 \leq n \leq g(a,b) = g
 \end{aligned}$$

$$\begin{aligned}
 n_2 \quad n_1 & \quad n_1 + n_2 \in N_0 \quad n_1 + n_2 = g \quad . \\
 g(a,b) = g & \\
 n_1 & \quad . \\
 n_1 & \quad . \quad n_1 = x'a + y'b \quad x', y' \in \mathbb{Z} \quad . \\
 & \quad ;
 \end{aligned}$$

$$\begin{aligned}
 n_1 = xa + yb \quad , \quad 0 \leq x \leq b-1 \\
 . y \leq -1 & \quad N_0 \quad n_1 \\
 g = g(a,b) = ab - a - b.
 \end{aligned}$$

$$\begin{aligned}
 n_2 = g - n_1 = ab - a - xa - yb = (b-1-x)a + (-y-1)b, \\
 . \quad N_0 \quad n_2 \quad b \quad a \\
 g = g(a,b)
 \end{aligned}$$

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$$g = ab - a - b = (a-1)(b-1) - 1$$

$$(a,b) = 1$$

$$0 \leq n_1 \leq \frac{g-1}{2} \quad g = n_1 + n_2$$

$$g \geq n_2 \geq g - \frac{g-1}{2} = \frac{g+1}{2}$$

$$n_i \in \mathbf{N}_0 \quad g = n_1 + n_2 \quad \{n_1, n_2\}$$

$$n_i \geq \frac{g+1}{2} \quad n_2 \quad n_1$$

$$n_1 \leq \frac{g-1}{2}$$

$$0 \leq n_1 \leq \frac{g-1}{2} \quad n_1 \quad n_2 \geq \frac{g+1}{2}$$

$$n_2 \quad (n_i \in N_0) \quad g = n_1 + n_2 \quad \{n_1, n_2\}$$

$$\frac{g+1}{2}$$

$$\frac{g(a,b)+1}{2}$$

$$(a,b,c) = 1$$

$$c \quad b \quad a$$

$$a + c = db \quad (a,b,c) = 1$$

$$c \quad b \quad a \quad \dots$$

$$(b,c) = 1 \quad (a,b) = 1$$

$$x, y, z \in \mathbf{Z}$$

$$a + c = db \quad (a,b,c) = 1 \quad \dots$$

$$g = xa + yb + zc$$

\mathbf{Z}

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$$x', y', z' \in \mathbb{Z}$$

$$g = x'a + y'b + z'c.$$

$$\mathbb{Z}^* \mathbb{Z}^* \mathbb{Z}$$

$$t \in \mathbb{Z}$$

$$(x, y, z)T_1(t) = (x - bt, y + at, z)$$

$$(x, y, z)T_2(t) = (x, y + ct, z - bt)$$

$$(x, y, z)T_3(t) = (x - t, y + dt, z - t)$$

$$g = xa + yb + zc$$

$$a + c = bd$$

$$(x - t)a + (y + dt)b + (z - t)c = xa + yb + cz = g,$$

t

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$$g \in \mathbb{N}_0 \quad \{a, b, c\} \quad g \in \mathbb{N}_0$$

$$\{b, c\} \quad \{a, b\}$$

$$x \leq z \quad x, y, z \in \mathbb{N}_0 \quad g = xa + yb + zc$$

 $T_3(x)$

$$(x, y, z)T_3(x) = (0, y + dx, z - x)$$

$$g = (y + dx)b + (z - x)c$$

$$T_3(z) \quad z \leq x$$

$$g = (x - z)a + (y + dz)b,$$

$$\mathbb{N}_0 \quad g \in \mathbb{N}_0$$

$$\{b, c\} \quad \{a, b\}$$

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$$(a,b,c)=1 \quad \{a,b,c\} \quad \dots$$

$$: \quad \dots \quad \cdot a+c=db$$

$$g = g(a,b,c) = \max\{g_1, g_2\},$$

$$g_1 = \left\lfloor \frac{a}{d} \right\rfloor c - b, \quad g_2 = \left\lfloor \frac{c}{d} \right\rfloor a - b$$

$$\{a,b,c\} \quad g \quad \dots$$

$$g \quad \dots \quad \mathbf{N}_0$$

$$\dots \quad \{b,c\} \quad \{a,b\}$$

$$\cdot g = g(a,b,c) = g_1 \quad g_1 \geq g_2$$

$$z = \left\lfloor \frac{a}{d} \right\rfloor \cdot y = -1 \quad g_1 = yb + zc$$

$$\left\lfloor \frac{a}{d} \right\rfloor \leq \frac{a}{d} < \frac{a+c}{d} = b.$$

$$T_2(t) \quad \dots \quad \{b,c\} \quad g_1$$

$$(0, -1, \left\lfloor \frac{a}{d} \right\rfloor) T_2(t) = (0, y', z')$$

$$t \geq 1 \quad y = -1 \quad \cdot y', z' \in \mathbf{N}_0 \quad g_1 = y'b + z'c$$

$$z' = \left\lfloor \frac{a}{d} \right\rfloor - bt \leq (b-1) - b = -1$$

$$\cdot \mathbf{N}_0 \quad c \quad b \quad g_1 \quad \cdot$$

$$(0, -1, \left\lfloor \frac{a}{d} \right\rfloor) T_3\left(\left\lfloor \frac{a}{d} \right\rfloor\right) T_1(-1) = \left(-\left\lfloor \frac{a}{d} \right\rfloor, -1 + d\left\lfloor \frac{a}{d} \right\rfloor, 0\right) T_1(-1)$$

$$= \left(b - \left\lfloor \frac{a}{d} \right\rfloor, d\left\lfloor \frac{a}{d} \right\rfloor - a - 1, 0\right).$$

$$g_1 = x_1 a + y_1 b$$

$$x_1 = b - \left\lfloor \frac{a}{d} \right\rfloor, \quad y_1 = d\left\lfloor \frac{a}{d} \right\rfloor - a - 1,$$

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$$0 \leq x' = b - z \leq \left\lfloor \frac{c}{d} \right\rfloor$$

$$y'b = n_0 - x'a > g_2 - \left\lfloor \frac{c}{d} \right\rfloor a = -b.$$

$$\mathbf{N}_0 \quad n_0 = x'a + y'b \quad y' \geq 0$$

$$\{17, 2, 7\}$$

$$d = 12 \quad 17 + 7 = 2 \times 12$$

$$g_2 \geq g_1$$

$$g(17, 2, 7) = g_1 = \left\lfloor \frac{a}{d} \right\rfloor c - b = \left\lfloor \frac{17}{12} \right\rfloor \times 7 - 2 = 7 - 2 = 5.$$

$$X = x_0 + x_1 + \dots + x_n$$

$$x_0, x_1, \dots, x_n \in \mathbf{N}$$

$$X, Y \in \mathbf{N}_0 \quad nX \geq Y \quad Y = x_1 + 2x_2 + \dots + nx_n$$

$$x_0, x_1, \dots, x_n \in \mathbf{N}$$

$$X = x_0 + x_1 + \dots + x_n \quad Y = x_1 + 2x_2 + \dots + nx_n$$

$$nX - Y = nx_0 + (n-1)x_1 + \dots + x_{n-1} \geq 0$$

$$n' \quad nX \geq Y \quad X, Y \in \mathbf{N}_0$$

$$n'X \geq Y, \quad n' \leq n$$

$$x_{n'-1} = n'X - Y \quad x_{n'} = Y - (n'-1)X \quad (n'-1)X < Y$$

$$i = (n'-1) \quad n' \quad x_i = 0 \quad 0 \leq i \leq n$$

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$$x_0 + x_1 + \dots + x_n = x_{n-1} + x_n = (n'X - Y) + (Y - (n'-1)X) = X$$

$$\begin{aligned} x_1 + 2x_2 + \dots + nx_n &= (n'-1)x_{n-1} + n'x_n \\ &= (n'-1)(n'X - Y) + n'(Y - (n'-1)X) \\ &= n'Y - (n'-1)Y = Y. \end{aligned}$$

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$$a, k \in \mathbf{N} \quad \dots$$

$$\mathbf{N}_0 \quad \{a, a+k, a+2k, \dots, a+nk\} \quad g \in \mathbf{N}$$

$$X, Y \in \mathbf{N}_0 \quad g = aX + kY \quad , \quad nX > Y$$

$$g \quad 0 \leq i \leq n \quad a_i = a + ik \quad .$$

$$x_0, x_1, \dots, x_n \in \mathbf{N}_0 \quad \{a_0, a_1, \dots, a_n\}$$

$$\begin{aligned} g &= a_0x_0 + a_1x_1 + \dots + a_nx_n = a_0x_0 + (a+k)x_1 + \dots + (a+nk)x_n \\ &= a(x_0 + x_1 + \dots + x_n) + k(x_1 + 2x_2 + \dots + nx_n) \\ &= aX + kY \end{aligned}$$

$$nX > Y \quad . \quad X, Y \in N_0$$

$$g = aX + kY \quad nX > Y \quad X, Y \in N_0$$

$$\begin{aligned} g &= aX + kY = a(x_0 + x_1 + \dots + x_n) + k(x_1 + 2x_2 + \dots + nx_n) \\ &= a_0x_0 + (a+k)x_1 + \dots + (a+nk)x_n \\ &= a_0x_0 + a_1x_1 + \dots + a_nx_n \end{aligned}$$

$$N_0 \quad \{a_0, a_1, \dots, a_n\} \quad g$$

$$(a, k) = 1 \quad \{a, a+k, a+2k, \dots, a+nk\}$$

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$$(a, k) = 1 \quad n \quad k \quad a \quad \dots$$

$$\{a, a+k, a+2k, \dots, a+nk\}$$

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$$g = g(a, a+k, \dots, a+nk) = \begin{cases} \frac{a-n-1}{n}a + k(a-1) & , \quad n|(a-1) \\ \left[\frac{a-1}{n} \right] a + k(a-1) & , \quad \text{otherwise} \end{cases}$$

$$.n \nmid (a-1)$$

$$nz < a-1$$

$$z = \left[\frac{a-1}{n} \right]$$

$$n(z+1) > a-1$$

$$g = aX + kY$$

g

$$nX < Y \quad X, Y \in N_0$$

Y

$$X', Y' \in \mathbf{Z}$$

$$X' = X - kt \quad , \quad Y' = Y + at$$

$$. g = aX' + kY'$$

$$nX' = nX - kt \leq nX < Y \leq at + Y = Y'.$$

z

$$Y = a-1 \quad X = z$$

$$nX = nz < a-1 = Y.$$

$$N_0 \quad n_0 > g$$

$$X, Y \in \mathbf{Z} \quad (a, k) = 1$$

$$n_0 = aX + kY$$

$$. 0 \leq Y \leq a-1$$

$$aX = n_0 - kY > g - kY \geq az,$$

$$() \quad . X \geq z+1 \quad X > z$$

$$nX \geq n(z+1) > a-1 \geq Y,$$

n_0

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$$z = \frac{a-1}{n} \quad n|(a-1)$$

$$g = (z-1)a + (a-1)k$$

$$g = aX + kY \quad X, Y \in \mathbb{N}_0$$

$$nX < Y$$

$$X = z-1 \quad , \quad Y = a-1$$

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$$nX = n\left(\frac{a-1}{n} - 1\right) = a-1-n < a-1 = Y.$$

g

$$n_0 > g$$

$$.0 \leq Y \leq a-1$$

$$aX = n_0 - kY > g - k(a-1) = a(z-1)$$

$$.a-1 \geq Y \quad nX \geq nz \quad .X \geq z \quad X > (z-1)$$

n_0

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$$n=1|(a-1) \quad \{a, a+k\}$$

$$g(a, a+k) = (a-2)a + k(a-1) = a^2 - 2a + ka - k$$

$$= a(a+k) - a - (a+k)$$

$$.g = a-1$$

$$k=1$$

$$g = k(a-1)$$

$$n > a-1$$

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